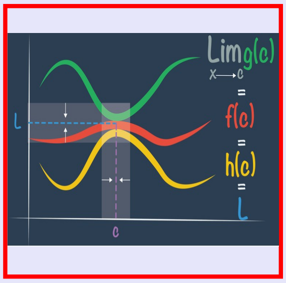
Math 261

Spring 2022

Lecture 4



Some review:

1) Show that $f(x)=2x^2-3x+4$ is cont. at $x=2$.

To be cont. at $x=a \Rightarrow \lim_{x\to a} f(x) = f(a)$

1) $f(2) = 2(2)^2-3(2)+4 = 2\cdot4-3\cdot2+4$
$= 8-6+4 = \boxed{6}$

2) $\lim_{x\to2} f(x) = \lim_{x\to2}\left[2x^2-3x+4\right] = \cdots\cdots$

$= 2\left[\lim_{x\to2} x\right]^2 - 3\left[\lim_{x\to2} x\right] + \lim_{x\to2} 4$

$= 2\cdot2^2-3\cdot2+4 = \boxed{6}$

3) Since $\lim_{x\to2} f(x) = f(2)$, then $f(x)$ is cont. at $x=2$.

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Suppose $f(x)=\begin{cases}\dfrac{x^2-9}{x+3} & x\neq -3\\ 6 & x=-3\end{cases}$

Is $f(x)$ cont. at $x=-3$?

1) $f(-3)=6$

2) $\lim\limits_{x\to -3} f(x)=\lim\limits_{x\to -3}\dfrac{x^2-9}{x+3}=\dfrac{0}{0}$ I.F.

$=\lim\limits_{x\to -3}\dfrac{(x+3)(x-3)}{x+3}=\lim\limits_{x\to -3}(x-3)$

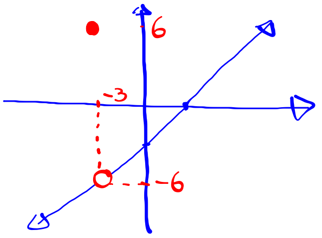
$=-3-3=\boxed{-6}$

3) $\lim\limits_{x\to -3} f(x)\neq f(-3)$

therefore $f(x)$ is not cont. at $x=-3$.

$f(x)=\dfrac{x^2-9}{x+3}=x-3$ if $x\neq -3$

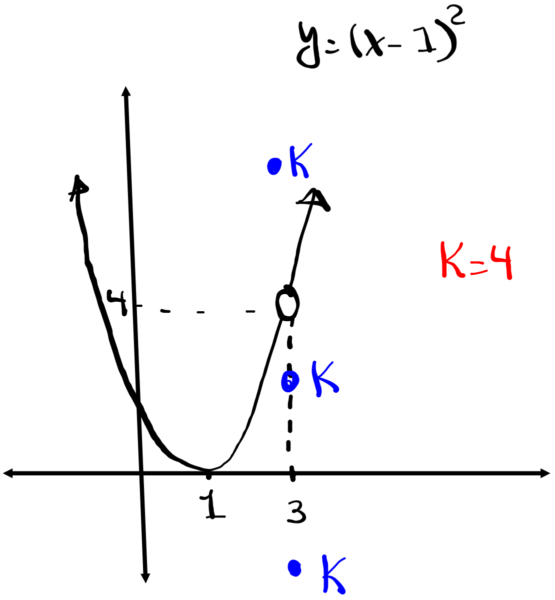
$f(x)=x-3$



Find K such that

$f(x)=\begin{cases}(x-1)^2 & \text{if } x\neq 3\\ K & \text{if } x=3\end{cases}$

is cont. at $x=3$.



K=4

$\lim\limits_{x\to 3} f(x)=(3-1)^2=4$

$f(3)=K$

$K=4\Rightarrow f(x)$ is cont. at $x=3$.

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Polynomial Functions are cont. everywhere.

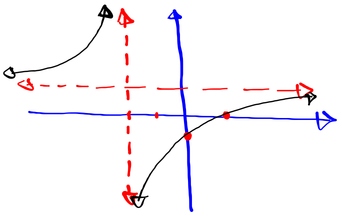
Rational Functions are continuous everywhere within the domain.

ex: Show $f(x)=\frac{x-1}{x+2}$ is cont. at $x=0$.

Rational Function

Domain $(-\infty,-2)\cup(-2,\infty)$

$x=0$ is in the domain ⇒ $f(x)$ is cont. at $x=0$



$f(0)=\frac{-1}{2}$

$\lim_{x\to 0} f(x)=\frac{-1}{2}$

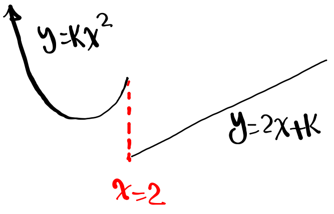
$\lim_{x\to 0} f(x)=f(0)$

∴ cont. at $x=0$.

Find K such that

$$f(x)=\begin{cases} Kx^2 & \text{if } x\le 2 \\ 2x+K & \text{if } x>2 \end{cases}$$

is cont. everywhere.



$\lim_{x\to 2^-} f(x)=K(2)^2=4K$

$\lim_{x\to 2^+} f(x)=2(2)+K=4+K$

For $x\le 2$

$f(x)=Kx^2$

Polynomial function

∴ Cont. everywhere.

For $x>2$

$f(x)=2x+K$

linear function

Polynomial function

∴ Cont. everywhere.

⇒ for $\lim_{x\to 2} f(x)$ exist,

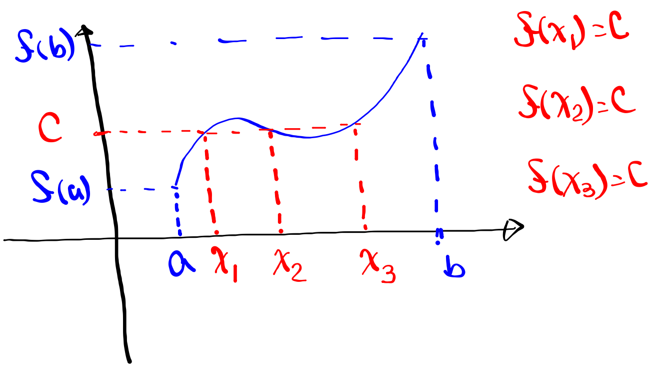
$4K=4+K$

$3K=4$

$\boxed{K=\frac{4}{3}}$

Intermediate Value Theorem:

If $f(x)$ is continuous on $[a,b]$, and C is a number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in $[a,b]$ such that $f(x)=C$.

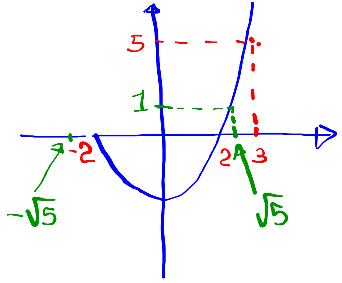


$f(x)=x^2-4$ on $[-2,3]$, show there there at least one number in $[-2,3]$ such that $f(x)=1$.

$x^2-4=1$

$x^2=5$

$x=\pm\sqrt{5}$



By I.V.T. $f(x)$ is cont on $[-2,3]$

$f(-2) \le 1 \le f(3)$

$0 \le 1 \le 5$

there is at least one number x in $[-2,3]$ such that $f(x)=1$

C

$-2<\sqrt{5}<3$

Show $\underbrace{x^3-4x+1}=0$ has at least one Solution in [1,2].

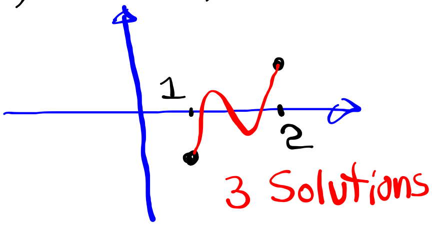
$f(x)=x^3-4x+1$

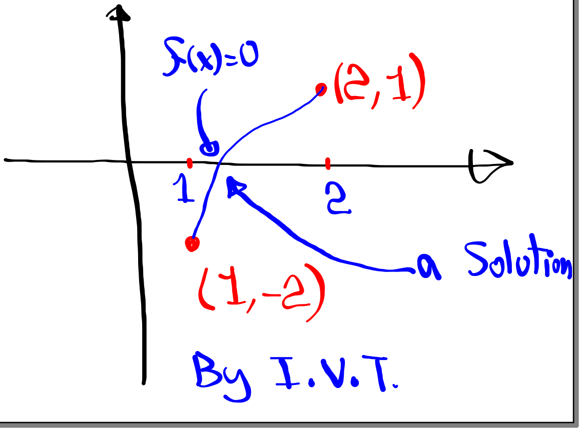
Poly nomial Function

$\therefore$ cont. $(-\infty,\infty)$

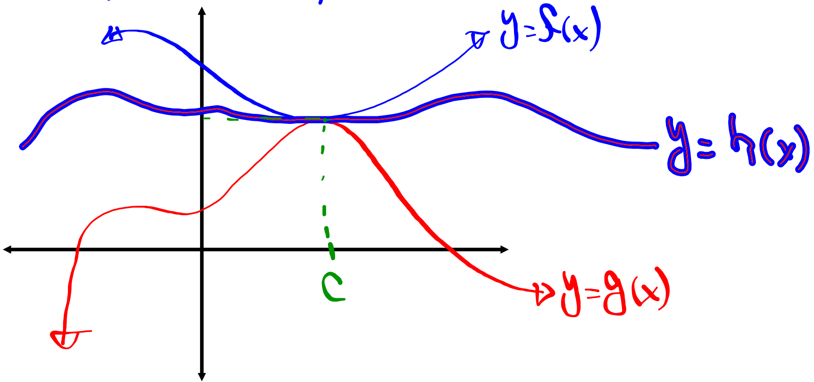
$f(1)=1^3-4(1)+1=-2$

$f(2)=2^3-4(2)+1=1$





Consider the graph below



If $g(x)\le h(x)\le f(x)$, and $\lim_{x\to c} f(x)=\lim_{x\to c} g(x)=L$

then $\lim_{x\to c} h(x)=L$

Squeeze Theorem

ex: Find $\lim_{x\to 0} x^2\sin^2\frac{1}{x}$

Can we plug in $x=0$ to evaluate the limit?

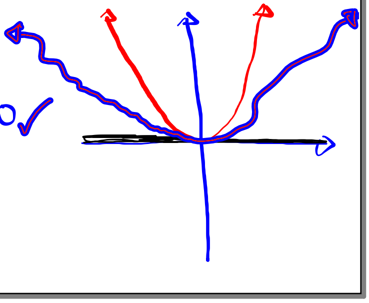
Recall from trig ⇒ $-1 \le \sin\square \le 1$

So $0 \le \sin^2\square \le 1$

multiply by x^2 ⇒ $x^2\cdot 0 \le x^2\sin^2\square \le x^2\cdot 1$

$x^2 \ge 0$

$0 \le x^2\sin^2\square \le x^2$



$\lim_{x\to 0} 0 = 0$ ✓

$\lim_{x\to 0} x^2 = 0$ ✓

⇒ $\lim_{x\to 0} x^2\sin^2\square = 0$ ✓

By S.T.

Given $\boxed{1-x^2} \le \boxed{f(x)} \le \boxed{\cos x}$ for all x in $(-\frac{\pi}{2},\frac{\pi}{2})$

∀

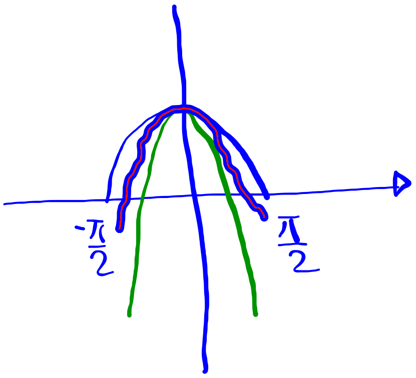
Find $\lim_{x\to 0} f(x)$

$\lim_{x\to 0}\cos x = 1$

$\lim_{x\to 0}(1-x^2) = 1$

⇒ $\lim_{x\to 0} f(x) = 1$

By S.T.



Find $$\lim_{x \to K} \frac{x^3 - Kx^2}{x^2 - K^2} = \frac{K^3 - K \cdot K^2}{K^2 - K^2} = \frac{0}{0} \text{ I.F.}$$

$$\lim_{x \to K} \frac{x^2 \cancel{(x-K)}}{\cancel{(x-K)}(x+K)} = \lim_{x \to K} \frac{x^2}{x+K}$$

$$= \frac{K^2}{K+K} = \frac{K^2}{2K} = \boxed{\frac{K}{2}}$$

Find $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for any $\underbrace{\text{quadratic function.}}_{f(x) = ax^2 + bx + c}$

$$f(x+h) = a(x+h)^2 + b(x+h) \cancel{+c}$$

$$f(x) = ax^2 + bx \cancel{+c}$$

$$f(x+h) - f(x) = a(x+h)^2 + b(x+h) - ax^2 - bx$$

$$= a[x^2 + 2xh + h^2] + b(x+h) - ax^2 - bx$$

$$= \cancel{ax^2} + 2axh + ah^2 \cancel{+bx} + bh \cancel{-ax^2} \cancel{-bx}$$

$$= 2axh + ah^2 + bh$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2axh + ah^2 + bh}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2ax + ah + b)}{\cancel{h}}$$

$$= \lim_{h \to 0} [2ax + \overset{\to 0}{\cancel{ah}} + b] = \boxed{2ax + b}$$

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Prove $\lim_{\underset{a}{x\to 2}}\left(\underbrace{3x-5}_{f(x)}\right)=\underset{L}{1}$

1) verify $\lim_{x\to 2}(3x-5)=1$ ✓

$\lim_{x\to 2}(3x-5)=3(2)-5=6-5=1$ ✓

2) Show that for every $\varepsilon>0$, there is a $\delta>0$ such that

$|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|3x-5-1|<\varepsilon$ whenever $|x-2|<\delta$

$|3x-6|<\varepsilon$

$|3(x-2)|<\varepsilon$

$3|x-2|<\varepsilon$

$|x-2|<\frac{\varepsilon}{3}$

Pick $\delta=\frac{\varepsilon}{3}$

If $\varepsilon=1 \Rightarrow \delta=\frac{1}{3}$

If $\varepsilon=.3 \Rightarrow \delta=\frac{.3}{3}=.1$

If $\varepsilon=3 \Rightarrow \delta=\frac{3}{3}=1$

Prove $\lim_{\underset{a}{x\to -2}}\left(\underbrace{4-3x}_{f(x)}\right)=\underset{L}{10}$ ✓

1) Verify $\lim_{x\to -2}(4-3x)=10$ ✓

$\lim_{x\to -2}(4-3x)=4-3(-2)=4+6=10$ ✓

2) Show that for any $\varepsilon>0$, there is a $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $|x-a|<\delta$

$|4-3x-10|<\varepsilon$ whenever $|x-(-2)|<\delta$

$|-3x-6|<\varepsilon$

$|-3(x+2)|<\varepsilon$

$|-3|\,|x+2|<\varepsilon$

$3|x+2|<\varepsilon$

$|x+2|<\frac{\varepsilon}{3}$

Pick $\delta=\frac{\varepsilon}{3}$

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Prove $\lim_{x \to 2} (x^2+3x) = 10$ ✓

$f(x) = x^2+3x$ $a = 2$ $L = 10$ ✓

1) verify $\lim_{x \to 2} (x^2+3x) = 10$ ✓

$\lim_{x \to 2} (x^2+3x) = 2^2+3(2) = 10$ ✓

2) Show for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $|x-a| < \delta$

$|x^2+3x-10| < \varepsilon$ $|x-2| < \delta$

$|(x+5)(x-2)| < \varepsilon$ If we wish $\delta \leq 1$

$|x+5||x-2| < \varepsilon$ then $|x-2| \leq 1$

Need to bound

$-1 \leq x-2 \leq 1$

to have x+5 in the middle, Add 7

$-1+7 \leq x-2+7 \leq 1+7$

$-8 \leq 6 \leq x+5 \leq 8$

$-8 \leq x+5 \leq 8$

$|x+5| \leq 8$

$8|x-2| < \varepsilon$

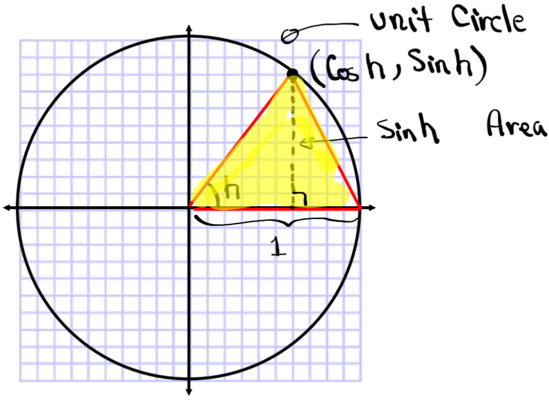
$|x-2| < \frac{\varepsilon}{8}$

$\delta = \frac{\varepsilon}{8}$

$\delta = \min\left\{1, \frac{\varepsilon}{8}\right\}$

If $\varepsilon = 1 \Rightarrow \delta = \min\left\{1, \frac{1}{8}\right\}$

If $\varepsilon = 16 \Rightarrow \delta = \min\left\{1, \frac{16}{8}\right\}$

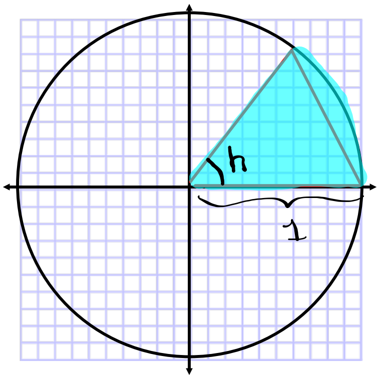


height

Area $= \frac{bh}{2}$

$= \frac{1 \cdot \sin h}{2}$

$= \frac{1}{2} \sin h$



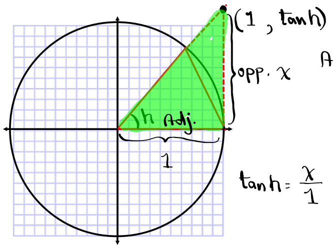
Area $= \frac{1}{2} r^2 h$

$= \frac{1}{2}(1)^2 \cdot h$

Area $= \frac{h}{2}$

Yellow area < blue Area

$\frac{1}{2} \sin h < \frac{1}{2} h$



height

$$\text{Area} = \frac{bh}{2}$$

$$= \frac{1 \cdot \tanh}{2}$$

$$\tan h = \frac{x}{1}$$

h is in QI

$$\frac{1}{2}\sin h < \frac{1}{2}h < \frac{1}{2}\tan h$$

$$\sin h < h < \tan h$$

$$\sin h < h < \frac{\sin h}{\cos h}$$

Divide by Sin h

$$\frac{\sin h}{\sin h} < \frac{h}{\sin h} < \frac{\sin h}{\cos h \sin h}$$

$$1 < \frac{h}{\sin h} < \frac{1}{\cos h}$$

Do reciprocal

$$\cos h < \frac{\sin h}{h} < 1$$

$$\lim_{h \to 0} 1 = 1$$

$$\lim_{h \to 0} \cos h = 1$$

$$\Rightarrow \boxed{\lim_{h \to 0} \frac{\sin h}{h} = 1}$$

By S.T.